Math 246B Lecture 18 Notes

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1 Hadamard Factorization

1.1 Lower bound on the product of Weierstrass factors

Let f be entire of finite order ρ, with zeros (a_k) such that $0 < |a_1| \leq a_2| \leq \cdots$. Let $m \in \mathbb{N}$ be such that $m \leq \rho < m+1$. Then we have the **Hadamard factorization**:

$$f(z) = e^{g(z)} z^p \prod_{k=1}^{\infty} E_m(z/a_k),$$

where g is entire, and p is the order of the zero at $z = 0$.

Theorem 1.1 (Hadamard)**.** *The function g is a polynomial of degree $\leq p$.*

We need a good lower bound on the canonical product away from the zeros $\{a_k\}$.

Proposition 1.1. *For any $s \in \mathbb{R}$ such that $\rho < s < m+1$, there is a constant $C_s = C > 0$ such that*

$$\left| \prod_{k=1}^{\infty} E_m(z/a_k) \right| \geq e^{-C|z|^s}$$

for all $z \in \mathbb{C} \setminus \bigcup D(a_k, |a_k|^{-m-1})$.

Proof. We need the following 2 estimates for $E_m(z)$:

1. $|E_m(z)| \geq e^{-C|z|^{m+1}}$ when $|z| < 1/2$: Write

   $$E_m(z) = (1-z)e^{\sum_{j=1}^{m} z/j} = e^w,$$

   where

   $$w = \log(1-z) + \sum_{j=1}^{m} \frac{z^j}{j} = -\sum_{j=m+1}^{\infty} \frac{z^j}{j}.$$

   So $|w| \leq 2|z|^{m+1}$, and the estimate follows.

2. $E_m(z)| \geq |1-z|e^{-C|z|^m}$ when $|z| > 1/2$: Write

$$|E_m(z)| \geq |1-z|e^{-|\sum_{j=1}^m z^j/j|},$$

where

$$\left|\sum_{j=1}^m \frac{z^j}{j}\right| \leq |z|^m \sum_{j=1}^m \frac{1}{|z|^{m-j}} \leq C|z|^m.$$

We write next

$$\prod_{j=1}^{\infty} E_m(z/a_k) = \underbrace{\prod_{|z/a_k|<1/2} E_m(z/a_k)}_{=A} \underbrace{\prod_{|z/a_k|\geq 1/2} E_m(z/a_k)}_{=B}.$$

The first estimate gives

$$|A| \geq \prod_{|z/a_k|<1/2} e^{-C|z/a_k|^{m+1}} = e^{-C|z|^{m+1}\sum_{|a_k|>2|z|} 1/|a_k|^{m+1}}.$$

Now if $\rho < s < m+1$, then $\sum 1/|a_k|^s < \infty$ (by the same argument as in last lecture). Then $|a_k|^{-m-1} = |a_k|^{-s}|a_k|^{s-m-1} \leq C|a_k|^{-s}|z|^{s-m-1}$, so we get the lower bound

$$|A| \geq e^{-C_s|z|^s}.$$

Next, the second estimate gives

$$|B| \geq \prod_{|z/a_k|>1/2} |1-z/a_k| \underbrace{\prod_{|z/a_k|\geq 1/2} e^{-C|z/a_k|^m}}_{=\exp(-C|z|^m\sum 1/|a_k|^m)}.$$

To bound this second term, we have $|a_k|^{-m} = |a_k|^{-s}|a_k^{s-m} \leq C|z|^{s-m}|a_k|^{-s}$, so

$$\prod_{|z/a_k|\geq 1/2} e^{-C|z/a_k|^m} \geq e^{-C_s|z|^s}.$$

Finally, using $|z-a_k| \geq 1/|a_k|^{m+1}$ for all k, we get

$$\prod_{|z/a_k|\geq 1/2} |1-a/z_k| \geq \prod_{|z/a_k|\geq 1/2} \frac{1}{|a_k|^{m+2}}.$$

Taking logs, we get

$$\sum_{|a_k|\leq 2|z|} (m+2)\log|a_k| \leq O(1)\log(2|z|) \underbrace{n(2|z|)}_{\leq C_\varepsilon|z|^{\rho+\varepsilon}} \leq O(1)|z|^s.$$

The result follows. $\square$

1.2 Proof of Hadamard's theorem

Let $\Omega = \mathbb{C} \setminus \bigcup D(a_k, 1/|a_k|^{m+1})$ be the domain from the previous proposition.

Proposition 1.2. *There exists a sequence $R_k \to \infty$ such that $\{|z| = R_k\} \subseteq \Omega$.*

Proof. Recall that $\sum_{k=1}^{\infty} 1/|a_k|^{m+1} < \infty$. Pick N so that $\sum_{k=N}^{\infty} 1/|a_k|^{m+1} < 1/2$. Set $A_k = \{x \in \mathbb{R} : |x - |a_k|| \leq |a_k|^{-m-1}\}$. Then $\sum_{k=N}^{\infty} < 1$. Given $L \in \mathbb{N}$ large, let $r \in [L_1, L+1] \setminus \bigcup_{k=N}^{\infty} A_k$; the set $\bigcup_{k=N}^{\infty} A_k$ has Lebesgue measure < 1. Then if $|z| = r$,

$$|z - a_k| \geq ||z| - |a_k|| \geq \frac{1}{|a_k|^{m+1}}.$$

If $L \geq L_0$ for large L_0, we also get

$$|z - a_k| \geq \frac{1}{|a_k|^{m+1}}$$

for $1 \leq k \leq N$, and the result follows. □

Now we can prove Hadamard's theorem. Recall that we have

$$f(z) = e^{g(z)} z^p \prod_{k=1}^{\infty} E_m(z/a_k).$$

Proof. When $|z| = R_j$, we have

$$|e^{g(z)}| = \frac{|f(z)|}{|z^p| \underbrace{\prod E_m(z/a_k)}_{\geq C_\varepsilon \exp(-|z|^{\rho+\varepsilon})}} \leq C_\varepsilon e^{|z|^{\rho+\varepsilon}}$$

for al $\varepsilon > 0$. By the Borel-Carathéodory estimate, which says

$$\sup_{|z|=r} |g(z)| \leq \frac{2r}{R-r} \sup_{|z|=R} \operatorname{Re}(g(z)) + \frac{R+r}{R-r}|g(0)|, \qquad r < R,$$

there exists a sequence $R_j \to \infty$ such that

$$|g(z)| \leq C_\varepsilon + |z|^{\rho+\varepsilon}, \qquad |z| = R_j, j = 1, 2, \ldots$$

By the usual Cauchy's estimates argument, g is a polynomial of degree $\leq \rho$. □